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Inversion Formulas arising in Inverse Boundary Value Problems

Masaru Ikehata (池畠優)
Department of Applied Physics, School of Engineering
Nagoya University,
Chikusa-Ku, Nagoya 464-01, Japan

1. Results. We formulate two inverse problems, which are analogus to the inverse conductivity problem [**10**].

Notation. Ω is a bounded domain of R^2 with smooth boundary $\partial\Omega$; ds is the standard measure on $\partial\Omega$; ν is the unite outer normal vector field on $\partial\Omega$; $X=\{H^{1/2}(\partial\Omega)\}^2$ and $Y=H^{3/2}(\partial\Omega)\times H^{1/2}(\partial\Omega)$; $B(X,X^*)$ is the Banach space of all bounded linear maps from X to its dual X^* and $B(Y,Y^*)$ that of all bounded linear maps from Y to its dual Y^*; $\nabla\mathbf{u}$ is the Jacobian matrix of a vector valued function $\mathbf{u}=\begin{pmatrix}u^1\\u^2\end{pmatrix}$ on Ω and $Sym\nabla\mathbf{u}$ its symmetric part; $\nabla^2 w$ is the Hessian matrix of a scalar function w on Ω; $\mathbf{a}\otimes\mathbf{b}=(a_ib_j)$ for two vectors $\mathbf{a}=(a_i)$, $\mathbf{b}=(b_j)$; $\mathbf{e}_1=\begin{pmatrix}1\\0\end{pmatrix}$, $\mathbf{e}_2=\begin{pmatrix}0\\1\end{pmatrix}$; $A_{11}=\mathbf{e}_1\otimes\mathbf{e}_1$; $A_{12}=\mathbf{e}_1\otimes\mathbf{e}_2+\mathbf{e}_2\otimes\mathbf{e}_1$; $A_{22}=\mathbf{e}_2\otimes\mathbf{e}_2$; $\partial_z=\frac{\partial}{\partial x_2}-z\frac{\partial}{\partial x_1}$ for each $z\in\mathrm{C}$.

Let $\mathbf{C}=(\mathbf{C}_{ijkl}(x))_{i,j,k,l=1,2}$ be a fourth-order tensor field over Ω with components $\mathbf{C}_{ijkl}\in L^\infty(\Omega)$. We denote by $\mathbf{C}(x)A$ the 2×2 -matrix $(\sum_{k,l}\mathbf{C}_{ijkl}(x)a_{kl})$ for each $x\in\Omega$ and 2×2-matrix $A=(a_{kl})$. We call $\mathbf{C}$ an elasticity tensor field if

$$\mathbf{C}_{ijkl}=\mathbf{C}_{klij}=\mathbf{C}_{lkij}$$

hold for each $i,j,k,l=1,2$ and there exists a positive number δ such that

$$\mathbf{C}(x)A\cdot A\equiv\sum\mathbf{C}_{ijkl}(x)a_{kl}a_{ij}\ge\delta|A|^2$$

holds for almost all $x\in\Omega$ and all real symmetric 2×2- matrix $A=(a_{ij})$.

For each elasticity tensor field $\mathbf{C}$ we define $\mathcal{L}_{\mathbf{C}}$, which is a second order system of partial differential operators acting $\{H^1(\Omega)\}^2$, via

$$\mathcal{L}_{\mathbf{C}}\mathbf{u}=\begin{pmatrix}\sum\frac{\partial}{\partial x_j}\{\mathbf{C}_{i1kl}(x)\frac{\partial u^k}{\partial x_l}\}\\ \sum\frac{\partial}{\partial x_j}\{\mathbf{C}_{i2kl}(x)\frac{\partial u^k}{\partial x_l}\}\end{pmatrix},\mathbf{u}=\begin{pmatrix}u^1\\u^2\end{pmatrix}\in\{H^1(\Omega)\}^2.$$

The associated Dirichlet-to-Neumann map $\Pi_{\mathbf{C}}\in B(X,X^*)$ is defined by

$$\Pi_{\mathbf{C}}(\varphi)=\{\mathbf{C}(x)Sym\nabla\mathbf{u}\}\nu|_{\partial\Omega},\varphi\in X,$$

where $\mathbf{u}\in\{H^1(\Omega)\}^2$ is the unique solution to

$$\mathcal{L}_{\mathbf{C}}\mathbf{u}=0\quad\text{in}\quad\Omega$$

$$\mathbf{u}|_{\partial\Omega} = \varphi.$$

$\Pi_{\mathbf{C}}(\varphi)ds$ is the force exerted across ds which deforms Ω into $\Omega + \mathbf{u}$.

On the other hand, for each elasticity tensor field $\mathbf{M}$ we define $L_{\mathbf{M}}$, which is a fourth-order partial differential operator acting on $H^2(\Omega)$, via

$$L_{\mathbf{M}}w = \sum \frac{\partial^2}{\partial x_i \partial x_j}\{\mathbf{M}_{ijkl}(x)\frac{\partial^2 w}{\partial x_k \partial x_l}\}, w \in H^2(\Omega).$$

The associated Dirichlet-to-Neumann map $\Pi^*_{\mathbf{M}} \in B(Y, Y^*)$ is defined by

$$\Pi^*_{\mathbf{M}}(\varphi) = \begin{pmatrix} -\{\frac{\partial}{\partial\tau}\mathbf{M}_\tau(w) + \mathbf{Q}(w)\} \\ \mathbf{M}_\nu(w) \end{pmatrix}|_{\partial\Omega}, \phi \in Y,$$

where $w \in H^2(\Omega)$ is the unique solution to

$$L_{\mathbf{M}}w = 0 \quad \text{in} \quad \Omega$$

$$\begin{pmatrix} w \\ \frac{\partial w}{\partial\nu} \end{pmatrix}|_{\partial\Omega} = \phi,$$

$$\mathbf{M}_\nu(w) = \mathbf{M}(x)\nabla^2 w \cdot \nu \otimes \nu, \mathbf{M}_\tau(w) = \mathbf{M}(x)\nabla^2 w \cdot \nu \otimes \tau,$$

$$\mathbf{Q}(w) = \sum \frac{\partial}{\partial x_\beta}\{(\mathbf{M}(x)\nabla^2 w)_{\alpha\beta}\}\nu_\alpha, \tau = \begin{pmatrix} -\nu_2 \\ \nu_1 \end{pmatrix}.$$

$\Pi^*_{\mathbf{M}}(\phi)$ is the external force applied to $\partial\Omega$ which deforms Ω into the graph of w; $\mathrm{M}_\nu(w)$ is the bending moment; the first component of $\Pi^*_{\mathbf{M}}(\phi)$ is the vertical reaction at $\partial\Omega$.

This talk is concerned with the following:

Inverse Problems.

I. Determine $\mathbf{C}$ from $\Pi_{\mathbf{C}}$;

II. Determine $\mathbf{M}$ from $\Pi^*_{\mathbf{M}}$.

The elasticity tensor field is said to be isotropic if there exist $\lambda, \mu \in L^\infty(\Omega)$, which are called the Lamé parameters, such that

$$\mathbf{C}(x)A = \lambda(x)Trace(A)I_2 + 2\mu(x)A$$

holds for almost all $x \in \Omega$ and all real symmetric 2×2-matrix A. Since isotropic $\mathbf{C}$ uniquely determines its Lamé parameters we write $\mathbf{C}_{(\lambda,\mu)}$ and $\Pi_{(\lambda,\mu)}$ instead of $\mathbf{C}$ and $\Pi_{\mathbf{C}}$, respectively.

The first problem for isotropic $\mathbf{C}$ was taken up by the author **[2]**, Akamatsu-Nakamura-Steinberg**[1]**, Nakamura-Uhlmann **[8]**. In particular, Nakamura-Uhlmann [8] proved that if λ and μ are smooth on $\overline{\Omega}$ and sufficiently close to constants, then $\Pi_{(\lambda,\mu)}$ uniquely determines (λ, μ). In **[9]** they treated the problem of determining $D^\alpha\mathbf{C}|_{\partial\Omega}$, $|\alpha| = 0, 1, \cdots$ from $\Pi_{\mathbf{C}}$ modulo smoothing operators on $\partial\Omega$, where $\mathbf{C}$ is not necessary isotropic and restricted to being in a class of anisotropic elasticity tensor fields, respectively.

The second problem for isotropic $\mathbf{M}$ was taken up by the author [3]. In [3] it is proved that if the Lamé parameters λ, μ of $\mathbf{M}$ are smooth and sufficiently close to constants on $\overline{\Omega}$, then $\Pi^*_{(\lambda,\mu)}$ together with $D^\alpha\lambda|_{\partial\Omega}$, $|\alpha| = 0, 1$ and $D^\beta\mu|_{\partial\Omega}$, $|\beta| = 0, 1, 2, 3$ uniquely determine (λ, μ).

In this talk first we shall point out that I and II are equivalent to each other on the simply connected Ω; second we consider the Fréchet derivative $d\Pi_{\mathbf{C}}$ and $d\Pi^*_{\mathbf{M}}$ at anisotropic $\mathbf{C}$ and $\mathbf{M}$, respectively; we shall study a relationship between them and give a characterization of the injectivity of $d\Pi_{\mathbf{C}}$ by the Stroh eigenvalues of $\mathbf{C}$.

For each elasticity tensor field $\mathbf{C}$ denote by $[\mathbf{C}]$ the symmetric 3×3-matrix

$$[\mathbf{C}] = \begin{pmatrix} \mathbf{C}_{1111} & \mathbf{C}_{1112} & \mathbf{C}_{1122} \\ \mathbf{C}_{1211} & \mathbf{C}_{1212} & \mathbf{C}_{1222} \\ \mathbf{C}_{2211} & \mathbf{C}_{2212} & \mathbf{C}_{2222} \end{pmatrix}.$$

We can define the transform $\mathbf{C}^*$ of $\mathbf{C}$ characterized by

$$[\mathbf{C}]^{-1} = PJ[\mathbf{C}^*]JP$$

where

$$J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

For the detail of the properties of this transform we refer the readers to [5] and [6]. It follows from the definition that $(\mathbf{C}^*)^* = \mathbf{C}$ and $(\mathbf{C}_{(\lambda,\mu)})^* = \mathbf{C}_{(\lambda^*,\mu^*)}$ with $\lambda^* = -\frac{\lambda}{4\mu(\lambda+\mu)}$, $\mu^* = \frac{1}{4\mu}$. We prove in §2

Theorem 1[6, Theorem A]. Let Ω be simply connected. Then

$$\Pi_{\mathbf{C}_1} = \Pi_{\mathbf{C}_2} \Longleftrightarrow \Pi^*_{\mathbf{C}_1^*} = \Pi^*_{\mathbf{C}_2^*}.$$

As a corollary we have immediately

Corollary 2. Let Ω be simply connected. Then

$$\Pi_{(\lambda_1,\mu_1)} = \Pi_{(\lambda_2,\mu_2)} \Longleftrightarrow \Pi^*_{(\lambda_1^*,\mu_1^*)} = \Pi^*_{(\lambda_2^*,\mu_2^*)}.$$

This connects the work done by Nakamura-Uhlmann [8] to that done by the author [3]. Theorem 1 shows the equivalence of I and II on any simply connected Ω.

The following is a linearized version of Theorem 1.

Theorem 3[6, Theorem C]. Let Ω be simply connected and $\mathbf{M} = \mathbf{C}^*$. Then $\ker d\Pi_{\mathbf{C}}$ is topologically linear isomorphic to $\ker d\Pi^*_{\mathbf{M}}$ under the relative topology from $L^\infty(\Omega)$.

In the theorem stated below it is not assumed that Ω is simply connected.

Theorem 4[6, Theorem D]. Let $\mathbf{C}$ be homogeneous and $\mathbf{M} = \mathbf{C}^*$. Then,

$$\ker d\Pi_{\mathbf{C}} = 0 \Longleftrightarrow \ker d\Pi^*_{\mathbf{M}} = 0 \Longleftrightarrow D(P_{\mathbf{M}}) \neq 0$$

where $D(P_{\mathbf{M}})$ is the discriminant of the polynomial

$$P_{\mathbf{M}}(\tau) = \mathbf{M}\begin{pmatrix} 1 \\ \tau \end{pmatrix} \otimes \begin{pmatrix} 1 \\ \tau \end{pmatrix} \cdot \begin{pmatrix} 1 \\ \tau \end{pmatrix} \otimes \begin{pmatrix} 1 \\ \tau \end{pmatrix};$$

there is an explicit formula of the left inverse of $d\Pi^*_{\mathbf{M}}$.

This is proved under $\mathbf{C}_{1112} = \mathbf{C}_{1222} = 0$ in [5] and therin we wrote down explicitly the left inverse of $d\Pi_{\mathbf{C}}$ for such $\mathbf{C}$ with $D(P_{\mathbf{C}^*}) \neq 0$; the author does not have the explicit formula of the left inverse $d\Pi_{\mathbf{C}}$ for general $\mathbf{C}$ with $D(P_{\mathbf{C}^*}) \neq 0$; the roots of the algebraic equation $P_{\mathbf{C}^*}(\tau) = 0$ are called the Stroh eigenvalues of $\mathbf{C}$(see [5]).

In the next section we will give the proofs of Theorems 1 $\sim$ 4.

2 **Proofs.** Throughout this section $(\cdot)_{,j}$ stands for partial differentiation with respect to x_j for each $j = 1, 2$.

Proof of Theorem 1. We study the relationship between three function spaces

$$\mathcal{P}_{\mathbf{C}} \equiv \{\mathbf{u} \in H^1(\Omega, \mathrm{C}^2) | \mathcal{L}_{\mathbf{C}}\mathbf{u} = 0 \quad \text{in} \quad \Omega\},$$

$$\mathcal{S}_{\mathbf{C}^{-1}} \equiv \{\mathbf{s} \in L^2(\Omega, Sym(\mathrm{C}^2)) | \sum_\beta \mathbf{s}_{\alpha\beta,\beta} = 0, 2(\mathbf{C}^{-1}\mathbf{s})_{12,12} = (\mathbf{C}^{-1}\mathbf{s})_{11,22} + (\mathbf{C}^{-1}\mathbf{s})_{22,11} \quad \text{in} \quad \Omega\}.$$

$$\mathcal{A}_{\mathbf{C}^*} \equiv \{w \in H^2(\Omega, \mathrm{C}) | L_{\mathbf{C}^*} = 0 \quad \text{in} \quad \Omega\}.$$

We can easily check that the map

$$f : \mathcal{P}_{\mathbf{C}} \ni \mathbf{u} \longmapsto \mathbf{s} = \mathbf{C} Sym \nabla \mathbf{u} \in \mathcal{S}_{\mathbf{C}^{-1}}$$

is well defined. On the other hand, for the check of the well definedness of the map

$$g : \mathcal{A}_{\mathbf{C}^*} \ni w \longmapsto \mathbf{s} = -J'\nabla^2 w J' \in \mathcal{S}_{\mathbf{C}^{-1}}, J' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

we needs the following

Lemma 1[6, Lemma A]. For any function w, put

$$\mathbf{s} = -J'\nabla^2 w J'.$$

Then

$$\sum_\beta \mathbf{s}_{\alpha\beta,\beta} = 0$$

and

$$(\mathbf{C}^{-1}\mathbf{s})_{11,22} + (\mathbf{C}^{-1}\mathbf{s})_{22,11} - 2(\mathbf{C}^{-1}\mathbf{s})_{12,12} = L_{\mathbf{C}^*} w.$$

We claim

Lemma 2[6, Lemma B]. Let Ω be simply connected. Then both f and g are surjective.

The proof of this lemma is based on two facts stated below.

Let $\mathbf{E} = (\mathbf{E}_{ij}(x))_{i,j=1,2}$ be a second-order symmetric tensor field on Ω. Then if

$$2\mathbf{E}_{12,12} = \mathbf{E}_{11,22} + \mathbf{E}_{22,11}$$

holds, there exists a vector valued function $\mathbf{u}$ such that

$$\mathbf{E} = Sym\nabla\mathbf{u},$$

and vice versa; the equation

$$\sum_\beta \mathrm{s}_{\alpha\beta,\beta} = 0$$

is equivalent to

$$d(\mathrm{s}_{11}dx_2 - \mathrm{s}_{12}dx_1) = 0, d(\mathrm{s}_{21}dx_2 - s_{22}dx_1) = 0.$$

Now we can give the proof of Theorem 1. Applying Green's theorem to $\Pi_{\mathbf{C}_1} = \Pi_{\mathbf{C}_2}$ and using Lemma 2, we obtain that

$$\Pi_{\mathbf{C}_1} = \Pi_{\mathbf{C}_2}$$

$$\Longleftrightarrow$$

$$\forall \mathbf{u}_j \in \mathcal{P}_{\mathbf{C}_j} \quad \int_\Omega (\mathbf{C}_1 - \mathbf{C}_2)Sym\nabla\mathbf{u}_1 \cdot Sym\nabla\mathbf{u}_2 dx = 0$$

$$\Longleftrightarrow$$

$$\forall \mathrm{s}_j \in \mathcal{S}_{\mathbf{C}_j^{-1}} \quad \int_\Omega (\mathbf{C}_2^{-1} - \mathbf{C}_1^{-1})\mathrm{s}_1 \cdot \mathrm{s}_2 dx = 0.$$

Here we note that for any $\mathbf{H} = (\mathbf{H}_{ijkl}(x))$ satisfying $\mathbf{H}_{ijkl} = \mathbf{H}_{klij} = \mathbf{H}_{klji}$ there exists a unique $\mathbf{H}^\dagger$ such that

$$[\mathbf{H}^\dagger] = J[H]J.$$

Then we have

$$\mathbf{H}\mathrm{s}_1 \cdot \mathrm{s}_2 = \mathbf{H}^\dagger\nabla^2 w_1 \cdot \nabla^2 w_2$$

for $\mathrm{s} = -J'\nabla^2 w_j J'$. Furthermore, we see that

$$(\mathbf{C}^{-1})^\dagger = \mathbf{C}^*.$$

Therefore $\Pi_{\mathbf{C}_1} = \Pi_{\mathbf{C}_2}$ is equivalent to

$$\forall w_j \in \mathcal{A}_{\mathbf{C}_j^*} \quad \int_\Omega \{(\mathbf{C}_2^{-1})^\dagger - (\mathbf{C}^{-1})^\dagger\}\nabla^2 w_1 \cdot \nabla^2 w_2 dx = 0$$

$$\Longleftrightarrow$$

$$\forall w_j \in \mathcal{A}_{\mathbf{C}_j^*} \quad \int_\Omega (\mathbf{C}_2^* - \mathbf{C}_1^*)\nabla^2 w_1 \cdot \nabla^2 w_2 dx = 0$$

$$\Longleftrightarrow$$

$$\Pi^*_{\mathbf{C}_1^*} = \Pi^*_{\mathbf{C}_2^*}. \qquad \text{Q.E.D.}$$

Since the proof of Theorem 3 can be done in the same way we omit the proof.

Proof of Theorem 4. At first we prove

Proposition 1 [4, Theorem A.] Let $\mathbf{M}$ be homogeneous. Then

$$\ker d\Pi^*_{\mathbf{M}} = \mathbf{O} \Longleftrightarrow D(P_{\mathbf{M}}) \neq 0;$$

there is an explicit formula of the left inverse of $d\Pi^*_{\mathbf{M}}$ for such $\mathbf{M}$.

By this proposition we see that the set of all homogeneous elasticity tensor fields is divided into two groups. This classification just coincides with that done by Lekhnitskii[7].

Proof of Proposition 1. We can write $P_{\mathbf{M}}(\tau)$ in the form

$$\mathbf{M}_{2222}(\tau-\alpha)(\tau-\overline{\alpha})(\tau-\beta)(\tau-\overline{\beta})$$

with some α, β satisfying $Im \quad \alpha \cdot Im \quad \beta > 0$. Hence

$$D(P_{\mathbf{M}}) \neq 0 \Longleftrightarrow \alpha \neq \beta$$

and $L_{\mathbf{M}}$ can be factorized as follows:

$$\mathbf{M}_{2222}\partial_\alpha \partial_{\overline{\alpha}} \partial_\beta \partial_{\overline{\beta}}.$$

Hence if $P_{\mathbf{M}}(z) = 0$, the function

$$exp\{-ic(x_1 + zx_2)\} \quad (c \in \mathrm{C})$$

is a solution of $L_{\mathbf{M}} w = 0$.

(i)$\Longleftarrow$

Assume $\alpha \neq \beta$. Let $\xi \in \mathrm{R}^2 \setminus \{0\}$ and

$$\{z_1, z_2\} = \{\alpha, \overline{\alpha}\}, \{\alpha, \overline{\beta}\}, \{\alpha, \beta\}, \{\beta, \overline{\beta}\}, \{\overline{\alpha}, \beta\}, \{\overline{\alpha}, \overline{\beta}\}.$$

Then

$$\mathbf{E}_\xi(x; z_1, z_2) := exp\{-i\frac{\xi_2 - z_1\xi_1}{z_2 - z_1}(x_1 + z_2 x_2)\}$$

is a solution of $L_{\mathbf{M}} w = 0$ and

$$\mathbf{E}_\xi(x; z_1, z_2)\mathbf{E}_\xi(x; z_2, z_1) = e^{-ix\cdot\xi}$$

holds. Let $d\Pi^*_{\mathbf{M}}(\mathbf{H}) = 0$. This is equivalent to

$$\int_\Omega \mathbf{H}(x)\nabla^2 u \cdot \nabla^2 v dx = 0$$

for any u, v; the solutions of $L_{\mathbf{M}} w = 0$ in Ω. Substitue $\mathbf{E}_\xi(x; z_1, z_2)$ and $\mathbf{E}_\xi(x; z_2, z_1)$ for u, v. Then we obtain

$$\mathbf{S}A\tilde{\mathbf{H}}(\xi) = 0$$

where

$$\mathbf{S} = \begin{pmatrix} 1 & \alpha+\overline{\alpha} & \alpha^2+\overline{\alpha}^2 & \alpha\overline{\alpha} & \alpha\overline{\alpha}^2+\alpha^2\overline{\alpha} & \alpha^2\overline{\alpha}^2 \\ 1 & \alpha+\overline{\beta} & \alpha^2+\overline{\beta}^2 & \alpha\overline{\beta} & \alpha\overline{\beta}^2+\alpha^2\overline{\beta} & \alpha^2\overline{\beta}^2 \\ 1 & \alpha+\beta & \alpha^2+\beta^2 & \alpha\beta & \alpha\beta^2+\alpha^2\beta & \alpha^2\beta^2 \\ 1 & \beta+\overline{\beta} & \beta^2+\overline{\beta}^2 & \beta\overline{\beta} & \beta\overline{\beta}^2+\beta^2\overline{\beta} & \beta^2\overline{\beta}^2 \\ 1 & \overline{\alpha}+\beta & \overline{\alpha}^2+\beta^2 & \overline{\alpha}\beta & \overline{\alpha}\beta^2+\overline{\alpha}^2\beta & \overline{\alpha}^2\beta^2 \\ 1 & \overline{\alpha}+\overline{\beta} & \overline{\alpha}^2+\overline{\beta}^2 & \overline{\alpha}\overline{\beta} & \overline{\alpha}\overline{\beta}^2+\overline{\alpha}^2\overline{\beta} & \overline{\alpha}^2\overline{\beta}^2 \end{pmatrix},$$

$$A\tilde{\mathbf{H}}(\xi) = \begin{pmatrix} \tilde{\mathbf{H}}A_{11}\cdot A_{11} \\ \tilde{\mathbf{H}}A_{11}\cdot A_{12} \\ \tilde{\mathbf{H}}A_{11}\cdot A_{22} \\ \tilde{\mathbf{H}}A_{12}\cdot A_{12} \\ \tilde{\mathbf{H}}A_{12}\cdot A_{22} \\ \tilde{\mathbf{H}}A_{22}\cdot A_{22} \end{pmatrix},$$

and

$$\tilde{\mathbf{H}}(\xi) = \int_\Omega e^{-ix\cdot\xi}\mathbf{H}(x)dx \quad (\xi \in \mathrm{R}^2).$$

Since

$$det \quad \mathbf{S} = -\{(\alpha-\overline{\alpha})(\alpha-\overline{\beta})(\alpha-\beta)(\overline{\alpha}-\overline{\beta})(\overline{\alpha}-\beta)(\overline{\beta}-\beta)\}^2 \quad ([4, \text{Lemma A}]),$$

we can conclude $A\tilde{\mathbf{H}}(\xi) = 0$ and it is possible to write down the left inverse of $d\Pi^*_{\mathbf{M}}(H)$ explicitly. The result is as follows. Put

$$C(\xi; \{z_1, z_2\}) := \{\frac{(z_2-z_1)^2}{(\xi_2 - z_1\xi_1)(\xi_2 - z_2\xi_1)}\}^2,$$

$$\mathbf{D}(\xi; \{z_1, z_2\}) := C(\xi; \{z_1, z_2\}) \int_{\partial\Omega} d\Pi^*_{\mathbf{M}}(\mathbf{H}) \begin{pmatrix} u|_{\partial\Omega} \\ \frac{\partial u}{\partial\nu}|_{\partial\Omega} \end{pmatrix} \cdot \begin{pmatrix} v|_{\partial\Omega} \\ \frac{\partial v}{\partial\nu}|_{\partial\Omega} \end{pmatrix} ds$$

for $u = \mathbf{E}_\xi(x; z_1, z_2)$ and $v = \mathbf{E}_\xi(x; z_2, z_1)$. Then

$$\mathbf{D}(0; \{z_1, z_2\}) := \lim_{\xi\to 0} D(\xi; \{z_1, z_2\})$$

exists and the left inverse of $d\Pi^*_{\mathbf{M}}$ is given by the following formula:

$$A\tilde{\mathbf{H}}(\xi) = \mathbf{S}^{-1}\mathbf{D}(\xi)$$

where

$$\mathbf{D}(\xi) := \begin{pmatrix} \mathbf{D}(\xi; \{\alpha, \overline{\alpha}\}) \\ \mathbf{D}(\xi; \{\alpha, \overline{\beta}\}) \\ \mathbf{D}(\xi; \{\alpha, \beta\}) \\ \mathbf{D}(\xi; \{\beta, \overline{\beta}\}) \\ \mathbf{D}(\xi; \{\overline{\alpha}, \beta\}) \\ \mathbf{D}(\xi; \{\overline{\alpha}, \overline{\beta}\}) \end{pmatrix}.$$

(ii)$\Longrightarrow$

Let $Im \quad z \neq 0$. We note that the identity

$$\nabla^2 w = \frac{1}{(\overline{z}-z)^2}\{\partial_{\overline{z}}^2 w A'_{11} + (-\partial_z\partial_{\overline{z}} w A'_{12} + \partial_z^2 w A'_{22}\} \quad ([4,\ 1.13])$$

holds for any scalar function w where

$$A'_{11} = \begin{pmatrix} 1 \\ z \end{pmatrix} \otimes \begin{pmatrix} 1 \\ z \end{pmatrix},$$

$$A'_{12} = \begin{pmatrix} 1 \\ z \end{pmatrix} \otimes \begin{pmatrix} 1 \\ \overline{z} \end{pmatrix} + \begin{pmatrix} 1 \\ \overline{z} \end{pmatrix} \otimes \begin{pmatrix} 1 \\ z \end{pmatrix},$$

$$A'_{22} = \begin{pmatrix} 1 \\ \overline{z} \end{pmatrix} \otimes \begin{pmatrix} 1 \\ \overline{z} \end{pmatrix}.$$

This yields

$$\mathbf{H}(x)\nabla^2 u \cdot \nabla^2 v = \frac{1}{(\overline{z}-z)^4}\sigma^t(\mathbf{H}A)\sigma \begin{pmatrix} \partial_{\overline{z}}^2 u \\ -\partial_z\partial_{\overline{z}} u \\ \partial_z^2 u \end{pmatrix} \cdot \begin{pmatrix} \partial_{\overline{z}}^2 v \\ -\partial_z\partial_{\overline{z}} v \\ \partial_z^2 v \end{pmatrix}$$

where

$$\sigma = \begin{pmatrix} 1 & 2 & 1 \\ z & z+\overline{z} & \overline{z} \\ z^2 & 2z\overline{z} & \overline{z}^2 \end{pmatrix},$$

$$\mathbf{H}A = \begin{pmatrix} \mathbf{H}A_{11}\cdot A_{11} & \mathbf{H}A_{11}\cdot A_{12} & \mathbf{H}A_{11}\cdot A_{22} \\ & \mathbf{H}A_{12}\cdot A_{12} & \mathbf{H}A_{12}\cdot A_{22} \\ & & \mathbf{H}A_{22}\cdot A_{22} \end{pmatrix} \in Sym(\mathrm{R}^3).$$

Now assume $z = \alpha = \beta$. If we take $\mathbf{H}$ such that

$$\sigma^t(\mathbf{H}A)\sigma = \begin{pmatrix} \partial_z^3\varphi & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \partial_{\overline{z}}^3\overline{\varphi} \end{pmatrix} (\Longleftrightarrow \mathbf{H}A = 2Re\{\partial_z^3\varphi \begin{pmatrix} \overline{z}^2 \\ -2\overline{z} \\ 1 \end{pmatrix} \otimes \begin{pmatrix} \overline{z}^2 \\ -2\overline{z} \\ 1 \end{pmatrix}\})$$

with some φ satisfying $D^\alpha\varphi = 0$ for $|\alpha| = 0, 1, 2$ on $\partial\Omega$, integration by parts tells us that

$$\int_\Omega \mathbf{H}(x)\nabla^2 u \cdot \nabla^2 v dx = \frac{1}{(\overline{z}-z)^4}\int_\Omega (\partial_z^3\varphi\partial_{\overline{z}}^2 u\partial_{\overline{z}}^2 v + \partial_{\overline{z}}^3\overline{\varphi}\partial_z^2 u\partial_z^2 v)dx$$

$$= \frac{2}{(\overline{z}-z)^4}\int_\Omega (\partial_z\varphi\partial_z\partial_{\overline{z}}^2 u\partial_z\partial_{\overline{z}}^2 v + \partial_{\overline{z}}\overline{\varphi}\partial_{\overline{z}}\partial_z^2 u\partial_{\overline{z}}\partial_z^2 v)dx = 0,$$

where $\partial_z^2\partial_{\overline{z}}^2 u = \partial_z^2\partial_{\overline{z}}^2 v = 0$ in Ω. Since $L_{\mathbf{M}} = \mathbf{M}_{2222}\partial_z^2\partial_{\overline{z}}^2$, we have $\mathbf{H} \in \ker d\Pi_{\mathbf{M}}^*$. Q.E.D.

In [4, Proposition A], we gave how to find such φ appeared above for each $\mathbf{H} \in \ker d\Pi_{\mathbf{M}}^*$ when Ω is simply connected.

Proposition 2 [**5, Theorem B.**] Let **C** be homogeneous. Then

$$D(P_{\mathbf{C}^*}) = 0 \Longrightarrow \ker d\Pi_{\mathbf{C}} \neq 0.$$

Proof. By assumption, we can write $P_{\mathbf{C}^*}(\tau)$ in the form

$$\mathbf{C}^*_{2222}(\tau - z)^2(\tau - \overline{z})^2$$

with $Im \quad z \neq 0$. Then we can show that

$$\mathrm{K}_z := \{\mathbf{H}|\sigma^t(\mathbf{H}A)\sigma = \begin{pmatrix} \partial_z^3\varphi & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \partial_{\overline{z}}^3\overline{\varphi} \end{pmatrix}, D^\alpha\varphi|_{\partial\Omega} = 0 \quad \text{for} \quad |\alpha| = 0,1,2\} \subset \ker d\Pi_{\mathbf{C}}$$

and

$$\ker d\Pi_{\mathbf{C}} \subset \ker d\Pi^*_{\mathbf{C}^*}.$$

These are proved as follows.
First we show

$$\mathbf{H} \in \ker d\Pi_{\mathbf{C}}$$

if and only if

$$\int_\Omega \mathbf{H}(x) Sym\nabla\mathcal{F}_{\mathbf{C}}\mathbf{w}_1 \cdot Sym\nabla\mathcal{F}_{\mathbf{C}}\mathbf{w}_2 dx = 0$$

for any $\mathbf{w}_j = (w_j^1, w_j^2)^t$ satisfying $L_{\mathbf{C}^*}\mathbf{w}_j = 0$ in $\Omega(j = 1, 2)$, where

$$\mathcal{L}_{\mathbf{C}}\mathcal{F}_{\mathbf{C}} = \mathcal{F}_{\mathbf{C}}\mathcal{L}_{\mathbf{C}} \sim L_{\mathbf{C}^*}\mathbf{I}_2.$$

Second we write **C** in terms of z in the form

$$\mathbf{C} \sim \mathbf{C}_z(\theta) \quad ([\mathbf{5}, \text{Proposition } 3]),$$

where

$$A\mathbf{C}_z(\theta) \sim \begin{pmatrix} t^2 \\ -st \\ 2t(\theta - \frac{1}{2}) \\ 4t(1 - \theta + \frac{s^2}{4t}) \\ -s \\ 1 \end{pmatrix},$$

$$t = z \cdot \overline{z}, s = z + \overline{z}, \frac{s^2}{4t} < \theta < 1.$$

Note that we ignored the nonzero constant multiplication factor.
Third we show that for any $\mathbf{w} = (w^1, w^2)$, the factorization

$$Sym\nabla\mathcal{F}_{\mathbf{C}}\mathbf{w} \sim \partial_{\overline{z}}^2 u A'_{11} - \frac{1}{2}\partial_z\partial_{\overline{z}}(u + v)A'_{12} + \partial_z^2 v A'_{22} \quad ([\mathbf{5}, \text{Proposition } 5])$$

holds where
$$\begin{pmatrix} u \\ v \end{pmatrix} \sim \begin{pmatrix} \theta - \frac{s^2}{4t} & 2-\theta-\frac{s^2}{4t} \\ -(2-\theta-\frac{s^2}{4t}) & -(\theta-\frac{s^2}{4t}) \end{pmatrix} \begin{pmatrix} \partial_{\overline{z}} w_z \\ \partial_z w_{\overline{z}} \end{pmatrix},$$
$$\begin{pmatrix} w_z \\ w_{\overline{z}} \end{pmatrix} = \begin{pmatrix} 1 & z \\ 1 & \overline{z} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$

Fourth we show that $\mathbf{H} \in \ker d\Pi_{\mathbf{C}}$ is equivalent to

$$\int_\Omega \sigma^t(\mathbf{H}A)\sigma \begin{pmatrix} \partial_{\overline{z}}^2 u \\ -\frac{1}{2}\partial_z\partial_{\overline{z}}(u+v) \\ \partial_z^2 v \end{pmatrix} \cdot \begin{pmatrix} \partial_{\overline{z}}^2 u' \\ -\frac{1}{2}\partial_z\partial_{\overline{z}}(u'+v') \\ \partial_z^2 v' \end{pmatrix} dx = 0 \quad ([\mathbf{5}, \text{Proposition } 6])$$

for any u, u', v, v'; the solutions of $L_{\mathbf{C}^*}w = 0$ in Ω. From this we obtain immediately $\mathrm{K}_z \subset \ker d\Pi_{\mathbf{C}}$. Finally put $u = v$ and $u' = v'$. Then we obtain $\ker d\Pi_{\mathbf{C}} \subset \ker d\Pi^*_{\mathbf{C}^*}$. Q.E.D.

Proposition 3 [6] Let $\mathbf{C}$ be homogeneous. Then

$$D(P_{\mathbf{C}^*}) \neq 0 \Longrightarrow \ker d\Pi_{\mathbf{C}} = \mathbf{O}.$$

Proof. Take a open ball B such that $\Omega \subset B$. Then

$$\ker d\Pi_{\mathbf{C}}(\text{on} \quad \Omega) \subset \ker d\Pi_{\mathbf{C}}(\text{on} \quad B)$$

by zero extension of $\mathbf{H} \in \ker d\Pi_{\mathbf{C}}(\text{on} \quad \Omega)$ outside Ω. Since B is simply connected, Theorem 2 and Proposition 1 imply

$$\ker d\Pi_{\mathbf{C}}(\text{on} \quad B) \simeq \ker d\Pi^*_{\mathbf{C}^*}(\text{on} \quad B) = \mathbf{O}$$

and hence $\ker d\Pi_{\mathbf{C}} = \mathbf{O}$ on Ω. Q.E.D.

This completes the proof of Theorem 4.

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